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TECHNICAL NOTES

Unsteady natural convection flow over a vertical plate embedded in a stratified medium

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INTRODUCTION

HEAT TRANSFER processes by natural convection in a stratified medium are frequently observed in our natural environment as well as in many technological applications. Stratification is very important in heat rejection to water bodies, such as lakes, rivers and the sea, from thermal sources, such as the condensers of power plants, and various industrial units. The problem of a vertical plate at uniform temperature immersed in a fluid whose temperature increases linearly with height, has been considered by Eichhorn [1], Chen and Eichhorn [2], and Venkatachala and Nath [3]. They used the series solution method, local nonsimilarity method and implicit finite difference method, respectively. A similarity solution for a natural convection flow over a heated isothermal wall suspended in a quiescent thermally stratified atmosphere has been obtained by Kulkarni *et al.* [4].

In many natural convection processes temporal variation in the surface temperature of the body arises. Such flows are quite common in environmental processes and many technological and industrial applications. In nuclear reactors, furnaces, electronic systems, etc., the start-up and shut-down involve consideration of natural convection flow transients. Natural convection problems with time-dependent surface temperature have been solved by Sparrow and Gregg [5] and by several others in the following years under various assumptions. Recently transient double diffusive natural convection flow in a stratified medium has been considered by Angirasa and Srinivasan [6].

Most of these studies are related to a step change in the surface temperature or in the heat flux input. A step change in the boundary condition may be unrealistic, with respect to actual physical circumstances. In the present paper solutions have been obtained for various imposed perturbations, continuously varying with time, over a basic steady surface temperature.

ANALYSIS

Consider a vertical plate situated in a stratified ambient fluid at temperature $T_\infty(x)$. The x-coordinate is measured from the leading edge of the plate and the y-coordinate is measured normally from the plate to the fluid. The wall is assumed to be of finite extent and its temperature varies with time. With the Boussinesq assumption, the governing boundary layer equations take the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

The initial conditions are

$$u(x,y,0)=u_i(x,y),\quad v(x,y,0)=v_i(x,y),\quad T(x,y,0)=T_i(x,y). \tag{4}$$

The boundary conditions are

$$\left.\begin{array}{ll} x>0: & \text{at } y=0 \quad u=v=0,\ T=T_w(t) \\ & \text{as } y\to\infty \quad u=0,\ T=T_\infty(x)=T_{\infty 0}+ax \\ & \text{where } a=\mathrm{d}T_\infty/\mathrm{d}x>0 \\ x\leqslant 0: & \text{for all } y \quad u=0,\ T=T_\infty(x). \end{array}\right\} \tag{5}$$

To render the equations dimensionless we have used the following transformations:

$$\xi=ax/\Delta T_0,\quad \eta=(y/x)(Gr_x/4)^{1/4},\quad t^*=(\nu/2x^2)Gr_x^{1/2}t,$$

$$\left.\begin{array}{c} \psi(x,y,t)=(64Gr_x)^{1/4}\nu F(\xi,\eta,t^*)\phi(t^*), \\ \phi(t^*)=(T_w-T_{\infty 0})/\Delta T_0,\quad G(\xi,\eta,t^*)=(T-T_\infty)/\Delta T_0, \\ \Delta T_0=T_{w0}-T_{\infty 0},\quad Gr_x=g\beta\Delta T_0 x^3/\nu^2, \\ T_\infty=T_{\infty 0}+ax,\quad a=\mathrm{d}T_\infty/\mathrm{d}x>0, \\ u=\partial\psi/\partial y,\quad v=-\partial\psi/\partial x \end{array}\right\} \tag{6}$$

where $a>0$ implies a stably stratified ambient fluid. The plate temperature T_w can be written as $T_{\infty 0}+(\Delta T_0)\phi$ consists of a basic steady distribution T_{w0} with a weak superimposed time varying distribution governed by the unsteady function $\phi(t^*)$. The introduction of the stream function ψ automatically satisfies the continuity equation. The governing equations and boundary conditions, using the above transformations, reduce to:

$$\begin{aligned} &F'''+3FF''\phi-2F'^2\phi-F'_{t^*}-F'\phi^{-1}\phi_{t^*}+2t^*F'F'_{t^*}\phi \\ &\quad+2t^*F'^2\phi_{t^*}-2t^*F_{t^*}F''\phi+2t^*FF''\phi_{t^*}+\phi^{-1}G \\ &\qquad=4\xi\phi(F'F'_\xi-F_\xi F'') \end{aligned} \tag{7}$$

$$\begin{aligned} &\phi^{-1}G''+3Pr\,FG'-4Pr\,\xi F'+2t^*G_{t^*}F'+2t^*F_{t^*}G' \\ &\quad-2t^*\phi^{-1}\phi_{t^*}FG'-\phi^{-1}G_{t^*}=4\xi\,Pr(F'G_\xi-G'F_\xi). \end{aligned} \tag{8}$$

Subscripts ξ and t^* denote derivatives with respect to them and prime denotes the derivative with respect to η. The boundary conditions are given by:

$$\left.\begin{array}{ll} \xi>0: & \eta=0,\ t^*\geqslant 0:\ F=F'=0,\ G=\phi-\xi \\ & \eta\to\infty,\ t^*\geqslant 0:\ F'=0,\ G=0 \\ \xi\leqslant 0: & \text{for all } \eta,\ t^*\geqslant 0:\ F'=0,\ G=0. \end{array}\right\} \tag{9}$$

The conditions at $t^*=0$ are given by the steady state equations obtained by putting $\phi=1$, $\phi_{t^*}=F_{t^*}=G_{t^*}=F'_{t^*}=0$ in (7) and (8). The skin friction coefficient C_f and local heat transfer coefficient Nu_x (Nusselt number), based on the initial

NOMENCLATURE

a ambient temperature gradient, $dT_\infty(x)/dx$
C_f local skin friction coefficient
F dimensionless stream function
$F''(\xi, 0, t^*)$ surface skin friction parameter
g gravitational acceleration
G dimensionless temperature
$G'(\xi, 0, t^*)$ surface heat transfer parameter
Gr_x local Grashof number
k thermal conductivity
L body height
Nu local Nusselt number based on the initial temperature difference, $T_{w0} - T_{\infty 0}$
$\overline{Nu}$, $\overline{Nu}_{ISO}$ average Nusselt number based on ΔT_m and L for the stratified and unstratified cases, respectively
Pr Prandtl number
q local heat transfer rate per unit area
S stratification parameter, $aL/\Delta T_m$
t, t^* dimensional and dimensionless times, repectively
T temperature
T_∞, $T_{\infty 0}$ ambient temperature and its value at $x = 0$, respectively
T_w, T_{w0} wall temperature and its value at $t = 0$, respectively
ΔT_0 initial temperature difference, $T_{w0} - T_{\infty 0}$
ΔT_m temperature difference, $T_{w0} - T_\infty$ at midheight of the body
x, y distances along and perpendicular to the surface.

Greek symbols
α thermal diffusivity
β bulk coefficient of thermal expansion
η, ξ transformed coordinates
μ, ν dynamic and kinematic viscosities, respectively
ρ density
τ shear stress at the surface
ϕ dimensionless unsteady function
ψ dimensional stream function.

Superscript
$'$ differentiation with respect to η.

Subscripts
ξ, t^* derivatives with respect to ξ and t^*, respectively
ISO, i isothermal medium and initial conditions, respectively.

temperature difference can be expressed as

$$\left.\begin{aligned} &C_f = \tau/\rho(\nu/x)^2 \\ &\text{where } \tau = \mu(\partial u/\partial y)_{y=0} \text{ or} \\ &C_f = 4(Gr_x/4)^{3/4}F''(\xi, 0, t^*)\phi(t^*) \\ &\text{and} \\ &Nu = qx/\Delta T_0 = -(Gr_x/4)^{1/4}G'(\xi, 0, t^*) \\ &\text{where } q = -k(\partial T/\partial y)_{y=0}. \end{aligned}\right\} \quad (10)$$

RESULTS AND DISCUSSION

The nonlinear coupled partial differential equations (7) and (8) under boundary conditions (9) and initial conditions obtained from (7) and (8) by putting $t^* = 0$, have been solved numerically using an implicit finite difference scheme in combination with the quasilinearization technique [7]. Quasilinearization is an extension of the Newton–Raphson approximation for the solution of differential equations. This method converts the nonlinear two-point boundary value problem into an iterative scheme of solution which evolves the step-by-step integration of linear differential equations with two-point boundary conditions. This method is described in complete detail in ref. [8].

The nonlinear partial differential equations (7) and (8) are first linearized using quasilinearization then resulting linear partial differential equations are expressed in difference form using the central difference formula in η-direction and backward difference formulae in ξ and t^*-directions. The equations are then reduced to a system of linear algebraic equations with a block tri-diagonal structure which is solved using Varga's algorithm [9]. The step-sizes $\delta\eta$, $\delta\xi$ and δt^* and the edge of boundary layer η_∞ have been optimized. Finally we have taken $\delta\eta = \delta\xi = \delta t^* = 0.05$ and η_∞ between 4 and 12 depending upon the values of the parameters. The results presented here are independent of the step-sizes at least up to the fourth decimal place.

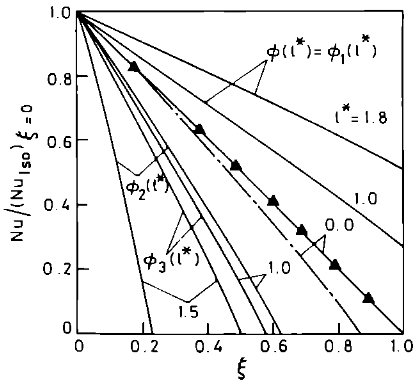

FIG. 1(a). Effect of wall temperature on local Nusselt number for $Pr = 6.0$. $\phi_1(t^*) = 1 + \varepsilon t^{*2}$, $\phi_2(t^*) = 1 - \varepsilon t^{*2}$, $\varepsilon = 0.25$; $\phi_3(t^*) = 1 + a(1 - e^{-ct^{*2}})$, $a = -0.5$, $c = 1.0$; ——, present result; – – –, series solution [1]; ▲, local nonsimilarity result [2].

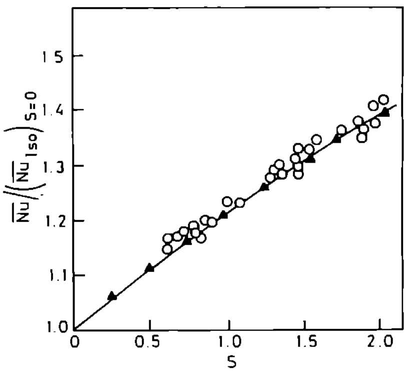

FIG. 1(b). Influence of stratification on steady state average Nusselt number. Theoretical predictions ($Pr = 6.0$) ——, present result; ▲, local nonsimilarity result [2]; ○, experimental data [2] ($5.5 < Pr < 7.5$, $1.7 \times 10^6 < Ra < 3.2 \times 10^7$).

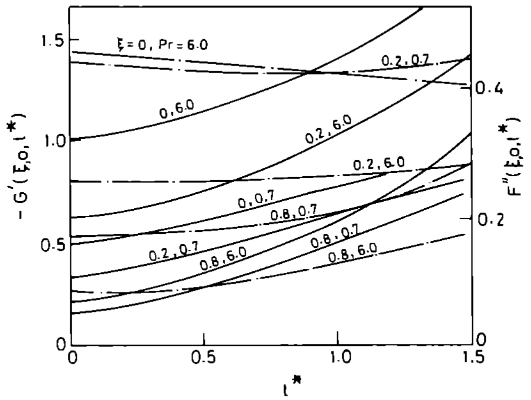

FIG. 2. Effect of wall temperature on heat transfer and skin friction parameters for $\phi(t^*) = 1+\varepsilon t^{*2}$, $\varepsilon = 0.25$. ——, $-G'(\xi, 0, t^*)$; — · —, $F''(\xi, 0, t^*)$.

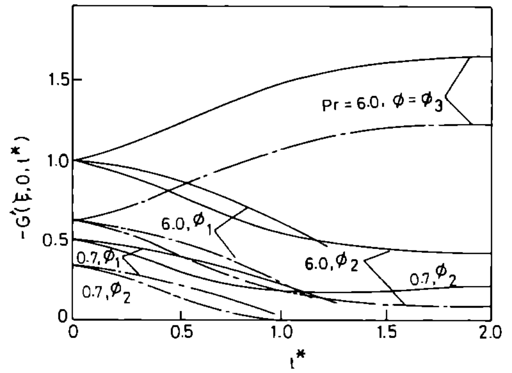

FIG. 3. Effect of wall temperature on heat transfer parameter for $\phi_1(t^*) = 1-\varepsilon t^{*2}$, $\phi_2(t^*) = 1+a(1-e^{-ct^{*2}})$, $a = -0.5$, $c = 1.0$ and $\phi_3(t^*) = 1+a(1-e^{-ct^{*2}})$, $a = 0.5$, $c = 1.0$. ——, $\xi = 0$; — —, $\xi = 0.2$.

The heat transfer results $Nu/(Nu_{ISO})_{\xi=0}$ for $Pr = 6.0$, showing the effect of the stratification are compared with the results of series solution [1] and local nonsimilarity method [2] in Fig. 1(a). In Fig. 1(b) the steady state result $\overline{Nu}/(\overline{Nu}_{ISO})_{S=0}$ is compared with the results of local non-similarity method and experimental data of ref. [2]. Here $\overline{Nu}$ is the average Nusselt number based on the temperature difference ΔT_m at midheight of the body and the subscript ISO refers to the isothermal medium. The ratio $\overline{Nu}/(\overline{Nu}_{ISO})_{S=0}$ for various linear stratifications represented by the parameter $S = aL/\Delta T_m$ is found in the same way as has been done in ref. [2]. To conserve space the details are not given here. In Table 1 we have compared our heat transfer results for $t^* = S = 0$ with the results of refs. [6, 10]. In all the above mentioned comparisons our results are in close agreement with the previous theoretical, as well as experimental, work.

Figure 1(a) shows that at a fixed x location local Nusselt number (Nu) based on the initial temperature difference (ΔT_0), decreases with the increase in stratification represented by the variable ξ. This figure also shows that on decreasing the wall temperature ($\phi = \phi_2$ and ϕ_3) the effect of stratification on the heat transfer becomes more pronounced. This is due to increase in the temperature difference between surface of the plate and environment which causes the increase in temperature gradient at the wall. The case of increasing wall temperature ($\phi = \phi_1$) is also shown in this figure.

The effects of Pr, stratification and variation in wall temperature on the heat transfer and the skin friction parameters are shown in Fig. 2. It is found that the effect of time-dependent wall temperature on the heat transfer parameter $-G'(\xi, 0, t^*)$ and skin friction parameter $F''(\xi, 0, t^*)$ is more pronounced for large values of Pr. For small values of ξ the skin friction parameter slightly decreases with the increase in wall temperature. This is perhaps due to the increase in the boundary layer thickness near the leading edge. At higher ξ locations buoyancy force increases with the increase of wall temperature which causes the increase in velocity and in skin friction parameter as shown in Fig. 2.

In Fig. 3 the results for quadratically decreasing ($\phi(t^*) = 1-\varepsilon t^{*2}$, $\varepsilon > 0$, $\varepsilon t^{*2} < 1$) and exponentially increasing and decreasing ($\phi(t^*) = 1 \pm a(1-e^{-ct^{*2}})$, $a > 0$, $c > 0$) wall temperature have been shown. The heat transfer parameter $-G'(\xi, 0, t^*)$ decreases rapidly with time when $\phi(t^*) = 1-\varepsilon t^{*2}$, $\varepsilon > 0$. For the exponential growth and decay in wall temperature $-G'(\xi, 0, t^*)$ attains a steady state after a certain time. To save space the behaviour of the skin friction parameter is not shown here because it does not give any new information. For higher values of stratification reversal in velocity profiles is found. Since it has been already predicted by previous investigators, the figures showing the velocity and temperature profiles are not given here.

Table 1. Comparison of the average Nusselt number for the case $t^* = s = 0$

$Gr_L\ Pr$	Churchill and Chu [10]	Angirasa and Srinivasan [6]	Present
0.7×10^6	15.531	16.923	16.653
7×10^6	32.175	30.623	31.047

CONCLUSIONS

The results are found to be strongly dependent on the variation of wall temperature and stratification. The reversal in velocity and temperature profiles is observed. At a particular height the local Nusselt number based on initial temperature difference decreases with the increase in stratification of the medium. The effect of stratification becomes more pronounced on decreasing the wall temperature. With the increase of wall temperature the skin friction parameter near the leading edge decreases but the skin friction parameter at higher locations and heat transfer parameter increases. These changes are more pronounced for large values of Prandtl number.

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Heat transfer from a stretching surface

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1. INTRODUCTION

THE HEAT transfer from a stretching surface is of interest in polymer extrusion processes where the object, after passing through a die, enters the fluid for cooling below a certain temperature. The rate at which such objects are cooled has an important bearing on the properties of the final product. In the cooling fluids the momentum boundary layer for linear stretching of sheet $U \propto x$ was first studied by Crane [1], whereas power law stretching $U \propto x^m$ was initially described by Afzal and Varshney [2].

Heat transfer from a linearly stretched surface $U \propto x$ based on the above work [1] has attracted the attention of several workers. The case of constant wall temperature has already been the subject of study [1, 3]. Similarly, for a non-uniform wall temperature closed form solution in terms of special functions has also been reported [5]. The case of uniform sheet velocity (zero stretching) is also well documented [7, 8].

The present work deals with heat transfer from an arbitrarily stretching surface $U \propto x^m$ for investigating the effects of non-uniform surface temperature. Several closed form solutions for specific values of m including their numerical solutions are presented in this technical note.

2. EQUATIONS

Let a polymer sheet emerging out of a slit at origin ($x = 0$) be moving with non-uniform velocity $U(x)$ in an ambient fluid at rest. The coordinate systems shown in Fig. 1, where coordinate x is the direction of motion of the sheet and y is the coordinate normal to it. The u and v are velocity components in the x and y directions, respectively. Further, ν is the molecular kinematic viscosity and σ the Prandtl number of the fluid. The boundary layer equations of mass, momentum and energy for two-dimensional constant pressure flow in usual notations are as follows:

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = \nu u_{yy} \tag{2}$$

$$uT_x + vT_y = \sigma^{-1}\nu T_{yy}. \tag{3}$$

The boundary conditions for the flow induced by stretching sheet (issuing from the slit $x = 0$) moving with non-uniform surface speed $U(x)$ in quiescent environment are:

$$y = 0,\quad u = U(x),\quad v = 0,\quad T = T_w(x) \tag{4}$$

$$y/\delta \to \infty,\quad u \to 0,\quad T \to T_\infty. \tag{5}$$

Introducing the similarity variables

$$\psi = \nu\sqrt{(2\xi)}f(\eta),\quad \eta = \frac{Uy}{\nu\sqrt{(2\xi)}},\quad \xi = \frac{Ux}{\nu(m+1)}$$

$$T = T_\infty + (T_w - T_\infty)\theta(\eta)$$

$$U = U_0 x^m,\quad T_w = T_\infty + Cx^n,\quad \beta = \frac{2m}{1+m} \tag{6}$$

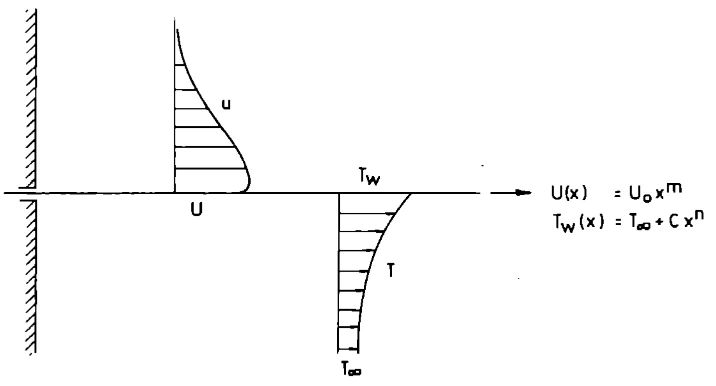

FIG. 1. Coordinate system for the flow induced by a polymer sheet moving with non-uniform surface speed in an ambient fluid at rest.